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Note

Iterative Solution of Homogeneous Integral Equations

A method is proposed for solving homogeneous Fredholm integral equations of the second kind which relies on rewriting such equations as equivalent inhomogeneous Fredholm integral equations of the second kind with "weaker" kernels. Then the usual techniques for solving inhomogeneous equations can be used in the present case. In particular a recently proposed method for iterative Neumann series solution of inhomogeneous equations appears to be natural and very suitable for this purpose. The method is illustrated numerically using the iterative Neumann series solution of the equivalent inhomogeneous equation.

1. Introduction

In this paper we develop a new method for the numerical solution of homogeneous Fredholm integral equations of the second kind [1, 2] for the unknown function $y(x)$

$$y(x) = \lambda \int_a^b K(x, z)\, y(z)\, dz, \qquad a \leqslant x \leqslant b, \tag{1.1}$$

or, in schematic operator notation,

$$y = \lambda K y, \tag{1.2}$$

where y is a real or complex function in $L^2(a, b)$ and K is a Fredholm kernel. As the equation is of the Fredholm type, the Fredholm alternative is valid and Eq. (1.1) can be uniformly approximated by a homogeneous matrix equation of finite rank, which can be solved by standard techniques. But in the case of a realistic problem the dimension of the resulting matrix equation could be large and an eigenfunction problem involving a large matrix is not a trivial numerical task.

Homogeneous equations of this type appear in various areas of physics so we present a general account of the method without considering a special problem of interest in physics. An interesting application could be the energy eigenfunction problem in quantum mechanics.

Here we rewrite Eq. (1.1) in the form of an equivalent inhomogeneous Fredholm integral equation of the second kind, which can be solved by standard methods. The equivalence between the homogeneous and the inhomogeneous equation is discussed. The equivalent inhomogeneous equation has a "weaker" kernel and a recently proposed method [3] for iterative solution of inhomogeneous equations appears to be attractive for our purpose. The method is illustrated numerically in few cases using

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the iterative Neumann series solution of the equivalent inhomogeneous equation. We also indicate how we can improve the convergence of the iterative scheme.

In Section 2 we present and discuss the method and illustrate it numerically. In Section 3 we present a brief summary and concluding remarks.

2. The Method

The method is based on the following simple algebraic manipulation. The equation we would like to solve is

$$y(x) = \lambda \int K(x, z)\, y(z)\, dz, \qquad a \leqslant x \leqslant b, \tag{2.1}$$

where in Eq. (2.1) and in the rest of the paper the integration limits are from a to b unless otherwise specified. Normalize $y(x)$ such that

$$\int \gamma(x)\, y(x)\, dx = 1, \tag{2.2}$$

where $\gamma(x)$ is an arbitrary function to be defined later. Using Eqs. (2.1) and (2.2) we have

$$y(x) = \lambda K(x, z_0) + \lambda \int K'(x, z)\, y(z)\, dz, \tag{2.3}$$

where

$$K'(x, z) = K(x, z) - K(x, z_0)\, \gamma(z), \tag{2.4}$$

and z_0 is an arbitrary chosen point in the interval (a, b). With the normalization given by Eq. (2.2) the solution $y(x)$ of Eq. (2.1) satisfies the equivalent inhomogeneous equation (2.3).

Note that Eq. (2.1) has solutions for certain selected values of λ, namely, when the operator λK has one of its eigenvalues equal to unity. Equation (2.3), being an inhomogeneous equation, has a unique solution for all λ unless its kernel has an eigenvalue equal to unity. An interesting question to ask at this stage is how, by solving Eq. (2.3), can we calculate the values of λ for which Eq. (2.1) has a solution and also find the solution of Eq. (2.1) when it exists. By construction all the solutions of Eq. (2.3) which satisfy Eq. (2.2) are solutions of Eq. (2.1). So one should solve Eq. (2.3) for a particular λ and check whether this solution also satisfies Eq. (2.2). If it does, then for this value of λ, Eq. (2.1) has a solution which is identical to the solution of Eq. (2.3). Alternatively if the value of λ for which Eq. (2.1) has a solution is given then for this value of λ the solution of Eq. (2.3) satisfies Eqs. (2.1) and (2.2).

Equation (2.3) can be solved by standard methods as the Fredholm alternative is valid. However, the form of Eq. (2.3) is such that the iterative methods formulated in Ref. [3] should be applicable. Let [3]

$$\gamma(z) = \frac{\int w(x)\, K(x, z)\, K(x, z_0)\, dx}{\int w(x)\, K(x, z_0)\, K(x, z_0)\, dx}, \tag{2.5}$$

and choose $w(x) = x^n$, where n is a small positive or negative integer provided that the integrals in Eq. (2.5) remain finite for such a choice.

In order to test the method we consider the numerical solution of

$$y(x) = \lambda \int_0^1 K(x, z)\, y(z)\, dz, \tag{2.6}$$

where K is assumed to have the three forms

$$K(x, z) = 0.1(x + z + 1)^5, \tag{2.7}$$

$$K(x, z) = (x + z)^6\, (x + z + 1)^{-1}, \tag{2.8}$$

and

$$K(x, z) = x^2 z + \frac{54}{59} x(1 + z). \tag{2.9}$$

The kernels given by Eqs. (2.7) and (2.8) were studied before in Ref. [3]. The kernel given by Eq. (2.9) is specially interesting because it involves the sum of two separable

TABLE I

Solution $y(x)$ of Eq. (2.6) for $\lambda = 0.15892$, $w(x) = x^2$ and $z_0 = 0.33187$ with K Defined by Eq. (2.7)

	$y(x)$		
	Solution of Eq. (2.3) after		Calculated from Eq. (3.1)
x	2 iterations	3 iterations	
0.99726	1.03766	1.03766	1.03764
0.93491	0.92524	0.92524	0.92522
0.79448	0.70774	0.70774	0.70772
0.58772	0.46418	0.46418	0.46417
0.33187	0.26134	0.26134	0.26134
0.04831	0.12714	0.12714	0.12714

Note. The second and the third columns exhibit the iterative solution of Eq. (2.3) after two and three iterations, respectively. The last column exhibits $y(x)$ calculated using the solution of Eq. (2.3) in the right hand side of Eq. (2.6). In this case using the solution of Eq. (2.3) the overlap integral of Eq. (2.2) was found to be equal to 0.99998.

terms and for $\lambda = 1.0$, Eqs. (2.6) and (2.9) have the analytic solution (apart from an arbitrary normalization)

$$y(x) = 4x^2 + 9x. \tag{2.10}$$

Note that if the kernel K contains only one separable term then with the choice of γ given by Eq. (2.5) the kernel K' of Eq. (2.4) is identically zero and Eq. (2.3) yields the exact solution without any iteration.

Approximate the integrals in all the equations by discrete sums using 16 point Gauss–Legendre quadratures between 0 and 1 and solve Eq. (2.3) by its convergent iterative series solution. Experimentation revealed that the "best" convergence was

TABLE II

Values as in Table I for $\lambda = 0.28654$, $z_0 = 0.50690$ with K Defined by Eq. (2.8)

	$y(x)$		
	Solution of Eq. (2.3) after		Calculated from Eq. (3.1)
x	2 iterations	3 iterations	
0.99726	1.23265	1.23265	1.23263
0.93491	1.02434	1.02434	1.02432
0.79448	0.65842	0.65842	0.65842
0.58772	0.31891	0.31892	0.31891
0.33187	0.11102	0.11102	0.11102
0.04831	0.02565	0.02565	0.02565

Note. In this case the overlap integral of Eq. (2.2) was equal to 0.99999.

TABLE III

Values as in Table I for $\lambda = 1$, $z_0 = 0.50690$ with $\dot{K}$ Defined by Eq. (2.9)

	$y(x)$		
	Solution of Eq. (2.3) after		Exact solution
x	2 iterations	3 iterations	
0.99726	1.89173	1.89173	1.89173
0.93491	1.73940	1.73940	1.73940
0.79448	1.41298	1.41298	1.41296
0.58772	0.97428	0.97428	0.97425
0.33187	0.50056	0.50056	0.50053
0.04831	0.06486	0.06486	0.06486

Note. The entry in the last column represents the exact analytic solution given by Eq. (2.10) normalized in such a way that for $x = 0.99726$ the entries in the third and the fourth columns are identical.

obtained for $w(x)=x^2$ and $z_0=0.33187$ for Eq. (2.7) and $w(x)=x^2$ and $z_0=0.50690$ for Eqs. (2.8) and (2.9). (The point z_0 is taken to be one of the integration mesh points.) Then solve Eq. (2.3) for a wide range of values of λ and verify whether this solution also satisfies Eq. (2.2). If it does then for this value of λ, Eq. (2.1) also has a solution which is identical to the solution $y(x)$ of Eq. (2.3). The values of λ so obtained for which Eq. (2.1) has a solution are given by: (a) $\lambda=0.15892$ for Eq. (2.7), (b) $\lambda=0.28654$ for Eq. (2.8), and (c) $\lambda=1.0$ for Eq. (2.9). (In this last case λ is analytically known and the solution of Eq. (2.6) is given by Eq. (2.10).) Next we verified if the solution of Eq. (2.3) satisfies Eq. (2.6) by substituting this solution on the right hand side of Eq. (2.6) and explicitly evaluating this term and comparing it with the solution of Eq. (2.3). Numerical results are displayed in Tables I–III, which are self-explanatory.

3. Discussion

The present method of solving a homogeneous Fredholm integral equation of the second kind is simple and a small amount of numerical work yields high precision results. In the present method the numerically tedious process of calculation of determinants and/or diagonalization of large matrices have been avoided.

The iterative solution of the equivalent inhomogeneous equation (2.3) is expected to converge for a proper choice of z_0 and γ. If the convergence of Eq. (2.3) is not satisfactory, we can, following Ref. [3], introduce an auxiliary equation with better convergence properties. The solution of (2.3) can then be expressed in terms of the solution of the auxiliary equation. The present method and the method of Ref. [3] should respectively be considered as viable alternatives for solutions of homogeneous and inhomogeneous Fredholm integral equations of the second kind.

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